Power Series Solutions and Frobenius method

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Northridge

Outline

- Review last week
- Power series solutions
  - General approach
  - Application
- Frobenius method
  - Basic process
  - Application to Bessel's Equation
- Review for midterm on Wednesday

Northridge 2

Review last week

- Systems of differential equations derived from physical problems involving different, interacting parts
- Showed how to convert two second order equations into one fourth order equation
  - Solved fourth order equation for one variable then used algebra for second
  - Fit initial conditions on both variables using both solutions

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Review Last Week II

- Solved system of linear first order differential equations

$$\frac{dy_i}{dt} + \sum_{j=1}^{n} a_{ij} y_j = r_i(t) \qquad i = 1, \ldots n$$

$$\frac{d\mathbf{y}}{dt} + \mathbf{A}\mathbf{y} = \mathbf{r}$$

- Solved in terms of eigenvalues, l, and eigenvector matrix, **X**, for **A**

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Review Last Week III

- Define new variable, **s** = $\mathbf{X}^{-1}\mathbf{y}$ (**y** = **Xs**)
- Transform original equation as follows

$$\mathbf{X}^{-1}\mathbf{X}\frac{d\mathbf{s}}{dt} + \mathbf{X}^{-1}\mathbf{A}\mathbf{X}\mathbf{s} = \mathbf{X}^{-1}\mathbf{r} \quad \Rightarrow \quad \frac{d\mathbf{s}}{dt} + \Lambda\mathbf{s} = \mathbf{X}^{-1}\mathbf{r} = \mathbf{p}$$

- Transformed equation is scalar equation whose solution is known

$$s_i = e^{-\lambda_i t}\left[\int e^{\lambda_i t} p_i dt + C_i\right] = C_i e^{-\lambda_i t} + q_i$$

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Review Last Week IV

- Definitions to convert $s_i = C_i e^{\lambda_i t} + q_i$ into matrix equation **s** = **E**(t)**C** + **q** (**E**(0) = **I**)

$$\mathbf{E}(t) = \begin{bmatrix} e^{-\lambda_1 t} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & e^{-\lambda_2 t} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & e^{-\lambda_3 t} & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & e^{-\lambda_n t} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ \vdots \\ C_n \end{bmatrix} \quad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ \vdots \\ q_n \end{bmatrix}$$

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Review Last Week V

- Convert matrix equation for **s** into matrix equation for **y** using **y** = **Xs**
- Apply initial condition that $\mathbf{y} = \mathbf{y}_0$ at t = 0, (where **E** = **I**,): $\mathbf{C} = \mathbf{X}^{-1}\mathbf{y}_0 - \mathbf{q}_0$
- Result: $\mathbf{y} = \mathbf{XE}\,[\mathbf{X}^{-1}\mathbf{y}_0 - \mathbf{q}_0] + \mathbf{Xq}$
- Homogenous (**q** = **0**): $\mathbf{y} = \mathbf{XEX}^{-1}\mathbf{y}_0$

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Review ODE Solution Basis

- A homogenous n^{th} order ODE has a basis of n linearly independent solutions
- $d^2y/dx^2 - k^2y = 0$ has the following possible solutions: e^{kx}, e^{-kx}, sinh(kx), and cosh(kx) but only two of these are linearly independent
- Have to find complete basis to be able to represent all possible initial or boundary conditions

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Power Series Solutions

- Look at following differential equation and proposed power series solution
- Requires p(x), q(x) and r(x) that can be expanded in power series about $x = x_0$

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

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Getting the Solutions

- Differentiate power series solution and substitute it into differential equation

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} \qquad \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

$$\sum_{n=0}^{\infty} n(n-1)a_n(x-x_0)^{n-2} + p(x)\sum_{n=0}^{\infty} n a_n (x-x_0)^{n-1} + q(x)\sum_{n=0}^{\infty} a_n (x-x_0)^n = r(x)$$

- Look at simple example with p(x) = r(x) = 0 and $q(x) = k^2$

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Getting the Solutions II

$$\sum_{n=0}^{\infty} n(n-1)a_n(x-x_0)^{n-2} + k^2\sum_{n=0}^{\infty} a_n (x-x_0)^n = 0$$

- In order to satisfy the power series equation $\Sigma_m c_m x^m = 0$, all $c_m = 0$
- Combine two sums into one with common limits and common powers of x
  - Let n = m + 2 in first sum
  - First two terms in this sum are zero
  - Rewrite first sum so that it has same limits as second sum (after dropping first two terms)

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Manipulating Summations

$$\sum_{n=0}^{\infty} n(n-1)a_n(x-x_0)^{n-2} = 0 + 0 + \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}$$

$$= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}(x-x_0)^m = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-x_0)^n$$

- First sum now has same $(x - x_0)^n$ factor and 0 to ∞ limits as second sum

$$\sum_{n=0}^{\infty} n(n-1)a_n(x-x_0)^{n-2} + k^2\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + k^2 a_n\right](x-x_0)^n = 0$$

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Making Coefficients Vanish

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+k^2a_n\right](x-x_0)^n=0$$

- Each coefficient of $(x - x_0)^n$ vanishes if $(n + 2)(n + 1)a_{n+2} + k^2a_n = 0$
- Gives recursion equation for a_n $\quad a_{n+2}=-\dfrac{k^2a_n}{(n+2)(n+1)}$

$$a_2=-\frac{k^2a_0}{2}\qquad a_4=-\frac{k^2a_2}{(4)(3)}=\frac{k^4a_0}{4\cdot3\cdot2}$$

Continued next slide

Want a_n as function of n

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Making Coefficients Vanish II

$$a_2=-\frac{k^2a_0}{2}\qquad a_4=-\frac{k^2a_2}{(4)(3)}=\frac{k^4a_0}{4\cdot3\cdot2}$$

- Continue recursion equation for a_{n+2} $\quad a_{n+2}=-\dfrac{k^2a_n}{(n+2)(n+1)}$

$$a_6=-\frac{k^2a_4}{(4+2)(4+1)}=-\frac{k^2}{(6)(5)}\frac{k^4a_0}{4\cdot3\cdot2}=-\frac{k^6a_0}{6\cdot5\cdot4\cdot3\cdot2}$$

- Propose general equation for a_n with even n $\quad a_n=\dfrac{(-1)^{n/2}k^na_0}{n!}$

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Check a_n Equation

- Write general a_n equation for a_{n+2} then check ratio a_{n+2}/a_n

$$a_n=\frac{(-1)^{n/2}k^na_0}{n!}\qquad a_{n+2}=\frac{(-1)^{(n+2)/2}k^{n+2}a_0}{(n+2)!}$$

$$\frac{a_{n+2}}{a_n}=\frac{\dfrac{(-1)^{(n+2)/2}k^{n+2}a_0}{(n+2)!}}{\dfrac{(-1)^{n/2}k^na_0}{n!}}=\frac{-k^2n!}{(n+2)!}=-\frac{k^2}{(n+2)(n+1)}$$

- Proposed a_n equation gives same result for a_{n+2}/a_n derived from power series

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Repeat Process for odd a_n

- All odd a_n proportional to a_1
- Original solution now has two series
  - Solutions are expected power series for sine and cosine
  - a_0 and a_1 chosen to fit initial conditions

$$y(x)=a_0\left[1-\frac{[k(x-x_0)]^2}{2!}+\frac{[k(x-x_0)]^4}{4!}-\cdots\right]$$
$$+a_1\left[k(x-x_0)-\frac{[k(x-x_0)]^3}{3!}+\frac{[k(x-x_0)]^5}{5!}-\cdots\right]$$

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Summary for $\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=r(x)$

- Write the solution for y(x) as a power series in unknown coefficients $\quad y(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$
- Differentiate the power series to get the derivatives required in the differential equation $\quad \frac{dy}{dx}=\sum_{n=0}^{\infty}na_n(x-x_0)^{n-1}\quad \frac{d^2y}{dx^2}=\sum_{n=0}^{\infty}n(n-1)a_n(x-x_0)^{n-2}$
- Get series for p(x), q(x), r(x) if required
- Substitute into differential equation

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Summary Continues

- Rewrite the resulting equation to group terms with common powers of $x - x_0$.
- Set the coefficients of each power of $x - x_0$ equal to zero giving an equation relating neighboring values of a_n
- Relate coefficients with higher sub-scripts to those with lower subscripts.
- Initial unknown coefficients, e.g., a_0, a_1, etc., are found from initial conditions

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Summary Concludes

- Examine equations relating neighboring coefficients and try to obtain a general equation for each an in terms of the unknown coefficients a_0, a_1, etc.
- Substitute the general expression for a_n into the original power series for y(x)
- This is the final power series solution

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Frobenius Method

- Applied to differential equation below around $x_0 = 0$ where b(x)/x and $c(x)/x^2$ make usual power series method inapplicable

$$\frac{d^2y(x)}{dx^2} + \frac{b(x)}{x}\frac{dy(x)}{dx} + \frac{c(x)}{x^2}y = 0$$

- Solution similar to previous power series (with $x_0 = 0$) except for x^r factor

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

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Frobenius Method II

- Differentiate proposed solution two times
- Get power series for b(x) and c(x)

$$\frac{dy}{dx} = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} \qquad \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$b(x) = \sum_{n=0}^{\infty} b_n x^n \qquad c(x) = \sum_{n=0}^{\infty} c_n x^n$$

- Substitute into original equation

$$\frac{d^2y(x)}{dx^2} + \frac{b(x)}{x}\frac{dy(x)}{dx} + \frac{c(x)}{x^2}y = 0$$

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Frobenius Method III

$$\frac{d^2y(x)}{dx^2} + \frac{b(x)}{x}\frac{dy(x)}{dx} + \frac{c(x)}{x^2}y = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} +$$

$$\frac{\sum_{n=0}^{\infty} b_n x^n}{x}\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \frac{\sum_{n=0}^{\infty} c_n x^n}{x^2}\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

- Manipulate to get single summation with common power of x in each term

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Frobenius Method IV

- Multiply result by x^2 and combine x and x^2 factors with x^{n+r} terms in sums

$$\overset{A}{\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}} + \left(\sum_{n=0}^{\infty} b_n x^n\right) \bullet B$$

$$\left(\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} c_n x^n\right)C\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0$$

- Expand series and multiply term by term to get first few terms in the series for the case where $b(x) = b_0$ and $c(x) = c_0$

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Frobenius Method V

$$A\; r(r-1)a_0x^r + (1+r)ra_1x^{r+1} + (2+r)(1+r)a_2x^{r+2} + \cdots$$

$$+ b_0ra_0x^r + [b_1ra_0 + b_0(1+r)a_1]x^{r+1} +$$

$$B\qquad [b_2ra_0 + b_1(1+r)a_1 + b_0(2+r)a_2]x^{r+2} + \cdots$$

$$C\; + c_0a_0x^r + [c_1a_0 + c_0a_1]x^{r+1} + [c_2a_0 + c_1a_1 + c_0a_2]x^{r+2} + \cdots = 0$$

- Coefficients of x^r term must vanish
  - $r(r-1)a_0 + b_0ra_0 + c_0a_0 = 0$
  - Do not want $a_0 = 0$
  - This requires $r(r-1) + b_0r + c_0 = 0$

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Frobenius Method V

- Equation $r(r-1) + b_0 r + c_0 = 0$ is known as indicial equation
- It is a quadratic equation giving two solutions for (index) r

$$r = \frac{1 - b_0 \pm \sqrt{(1-b_0)^2 - 4c_0}}{2}$$

- Choose higher value of r for first solution
- Second ODE solution depends on r values
  - Double root, roots differing by an integer, roots differing by a non-integer

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Frobenius Method VI

- First and second solutions $y_1(x)$ and $y_2(x)$
- First solution, all cases $y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$
- Root difference $r_1 - r_2$ not an integer $y_2(x) = x^{r_2} \sum_{n=0}^{\infty} A_n x^n$
- Double root $y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} A_n x^n$
- Roots differ by integer (k may be 0) $y_2(x) = k y_1(x)\ln(x) + x^{r_2} \sum_{n=0}^{\infty} A_n x^n$

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Frobenius Method VII

- Overall approach with this method
  - Convert b(x) and c(x) into power series if these are not simple terms
  - Find indicial equation roots r_1 and r_2
  - Apply power series analysis to find a_n coefficients in y_1 equation
  - Based on roots, determine second solution
  - Apply power series method to find A_n (and possibly k) in correct y_2 equation

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Bessel's Equation

- Arises in mechanical and thermal problems in circular geometries
- The value of ν is a known parameter
- Solve as example of Frobenius method

$$\frac{d^2 y(x)}{dx^2} + \frac{1}{x}\frac{dy(x)}{dx} + \frac{x^2 - \nu^2}{x^2} y = 0 \qquad \frac{dy}{dx} = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad \frac{d^2 y}{dx^2} = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

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Bessel's Equation II

- Plug solution and derivatives into Bessel's equation and rearrange

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \left(x^2 - \nu^2\right) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + (n+r) - \nu^2\right] a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)^2 - \nu^2\right] a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

$$= \sum_{n=0}^{\infty} \left[(n+r)^2 - \nu^2\right] a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$Both = a_0 x^{r+2} + a_1 x^{r+3} + a_2 x^{r+4} + \cdots$$

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Bessel's Equation III

- Final arrangement gets indicial equation

$$\sum_{n=0}^{\infty} \left[(n+r)^2 - \nu^2\right] a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = \left[(0+r)^2 - \nu^2\right] a_0 x^r$$

$$+ \left[(1+r)^2 - \nu^2\right] a_1 x^{1+r} + \sum_{n=2}^{\infty} \left[(n+r)^2 a_n - \nu^2 a_n + a_{n-2}\right] x^{n+r} = 0$$

- Indicial equation ($r^2 - \nu^2 = 0$) roots $r = \pm\nu$
  - Solution gives double root if ν = 0
  - Roots differ by an integer for integer ν
  - Roots do not differ by an integer for non-integer ν

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Bessel's Equation IV

- Continue next week after midterm
- Get series solutions for Bessel functions for three cases
  - Double root for $\nu = 0$
  - Roots differing by an integer
  - Non-integer roots
- Find two different series for any value of ν just like finding sine and cosine series in power series solution for $d^2y/dx^2 + k^2y = 0$

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Review for Midterm

- Vectors, matrices and determinants
  - Basic operations, particularly multiplication
  - Find determinants and matrix inverses
  - Vectors are linearly dependent if $\Sigma\alpha_i\mathbf{v}_{(i)} = \mathbf{0}$ with at least one $\alpha_i \neq 0$
  - A **basis set** for an n-dimensional vector space has n linearly independent vectors that can represent any vector in the space
- Gauss elimination process for solving equations determines linear dependence

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Midterm Review II

- Solutions to linear equations **Ax** = **b**
  - Unique if rank[**A b**] = rank **A** = $N_{unknowns}$
  - Infinite solutions if rank[**A b**] = rank **A** is less than number of unknowns
  - No solution if rank[**A b**] ≠ rank **A**
- Eigenvalues and eigenvectors: $\mathbf{Ax} = \lambda\mathbf{x}$
  - $\text{Det}(\mathbf{A} - \mathbf{I}\lambda) = 0$ gives eigenvalues
  - Solve $(\mathbf{A} - \mathbf{I}\lambda)\mathbf{x} = \mathbf{0}$ for components of each eigenvector (one component arbitrary)

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Midterm Review III

- Diagonalize a matrix: $\mathbf{\Lambda} = \mathbf{X}^{-1}\mathbf{AX}$
  - **X** is matrix of eigenvectors
  - **Λ** is diagonal matrix of eigenvalues
  - Works only if **X** has an inverse
- Special matrices
  - Unitary matrix columns have $[\mathbf{x}^*_{(i)}]\bullet\mathbf{x}_{(j)} = \delta_{ij}$
  - Orthogonal matrix columns have $[\mathbf{x}_{(i)}]\bullet\mathbf{x}_{(j)} = \delta_{ij}$
  - Hermitian matrix $\mathbf{A}^* = \mathbf{A}^T$
    - For Hermitian matrix $\mathbf{A}^{-1} = \mathbf{A}^T$

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Midterm Review IV

- First-order differential equations
  - Separable forms, e.g. $dy/dx = f(x)g(y)$
  - General linear equation $dy/dx + f(x)y = g(x)$ has solution $y = e^{-p}[C + \int e^p g(x)dx]$ where $p = \int f(x)dx$
  - Other separable forms
  - Solutions to $dy/dx = f(x,y)$ exist over a region about $x_0 < \min(a, b/K)$ where a,b are is x,y borders and $K = \max(|f|)$
  - Unique solution if $|\partial f/\partial x|$ is bounded

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Midterm Review V

- Second-order differential equations with constant coefficients: $d^2y/dx^2 + \alpha dy/dx + \beta y = r(x)$: find λ_1 and λ_2

$$\lambda_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2} \qquad \lambda_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2}$$

  - $r(x) = 0$ gives homogenous solution, y_H
    - For real λ_1 and λ_2, $y_H = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$
    - For real $\lambda_1 = \lambda_2 = \lambda$, $y_H = (C_1+C_2x)e^{\lambda x}$
    - For complex roots, $y = A\cos(\omega x) + B\sin(\omega x)$, where $\omega^2 = \beta - (\alpha/2)^2 = \beta - \alpha^2/4$
  - For $r(x) \neq 0$ $y = y_H + y_P$

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Midterm Review VI

- For nonhomogeneous solutions find solution $y = y_H + y_P$
- To get particular solution, y_P
  - Write form for y_P, based on form for r(x)
  - Substitute postulated y_P with unknown constant(s) into particular equation
  - Equate coefficients of like terms to find unknown constants
  - Use $y = y_H + y_P$ to find constants from homogenous solution from boundary values

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Midterm Exam

- Open book and notes, including homework solutions
- Make your own notes to use for exam
  - You are in trouble if you have to use the book on an open-book exam
- May be useful to have integral tables
- More credit given for showing how to obtain solution than for providing final details of algebra or arithmetic

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